Ozsvath

Heegard-Floer homology

$Y^3 \rightsquigarrow HF(Y^3)$ O-Szabó

X^4 Donaldson

$(X^4, g) \rightsquigarrow \mathcal{M}(X^4) \rightsquigarrow \#\mathcal{M}(X^4)$

anti-self-dual
Yang-Mills equ.

'94 Seiberg-Witten

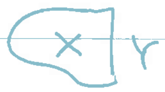
$(X, g) \rightsquigarrow \mathcal{M}(X^4) \rightsquigarrow \#\mathcal{M}(X^4)$

SW equ. Seiberg-Witten invariant

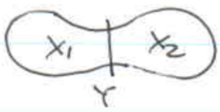
AIM : Define SW inv. in a combinatorial way

TQFT $Y \rightsquigarrow HF(Y) \rightarrow$ instanton Floer homology
$\rightarrow$ Monopole homology (SW Floer homology)

$Y_1 \boxed{W^4} Y_2 \rightsquigarrow F_W : HF(Y_1) \rightarrow HF(Y_2)$



$\mathcal{D}_X \in HF(Y)$



$\mathcal{D}_{X_1 \# X_2} = \langle \mathcal{D}_{X_1}, \mathcal{D}_{X_2} \rangle_{HF(Y)}$

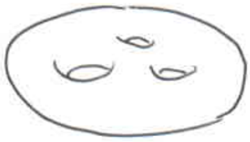
$Y \rightsquigarrow HF(Y)$

$K \subset Y \rightsquigarrow HFK(Y, K)$ O-Szabó Rasmussen
knot

Heegard Floer homology defined using holomorphic curves
Knot Floer techniques

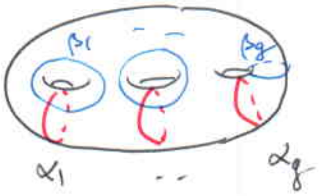
3 approaches to understand HFK in a comb. way

Y^3 = closed oriented 3-mfd



handle body

$$Y^3 = U_0 \cup_\Sigma U_1$$



g closed curves bounds disk in the handle

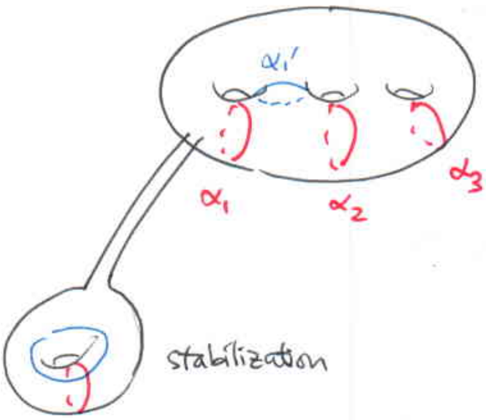
(homologically independent disjoint)

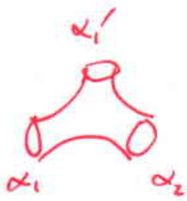
Ex. G_1

$S^1\times S^2$

$$(\Sigma_g, \{\alpha_1,\dots,\alpha_g\}, \{\beta_1,\dots,\beta_g\})$$

$\curvearrowleft$ Heegard diagram





handle slide

$$\left(\Sigma\ \begin{matrix}\alpha_1 \cdots \alpha_g \\ \beta_1 \cdots \beta_g\end{matrix}\right) \longrightarrow \left(\Sigma,\ \begin{matrix}\alpha_1', \alpha_2, \cdots \alpha_3 \\ \beta_1, \beta_2 \quad \beta_3\end{matrix}\right)$$

$\Sigma\setminus\{\alpha_1\dots\alpha_g, \beta_1\dots\beta_g\}$

$$(\Sigma_g, \{\alpha_1,\dots,\alpha_g\}, \{\beta_1,\dots,\beta_g\}, z)$$

$$\mathrm{Sym}^g\,\Sigma = \Sigma_g\times\cdots\times\Sigma_g / \mathfrak{S}_g$$

$\mathrm{Sym}^g(\Sigma)$: smooth alg. variety Σ : cpx str.

$\cup$

$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$ $\mathbb{T}_\beta$ similar totally real submfd

lagrangian w.r.t. a certain sympl. mfd

$\mathbb{T}_\alpha \subset \mathrm{Sym}^g(\Sigma)$
$\mathbb{T}_\beta \subset$ $\rightsquigarrow$

$CF(\mathbb{T}_\alpha, \mathbb{T}_\beta)$

work on $\mathbb{Z}/2$

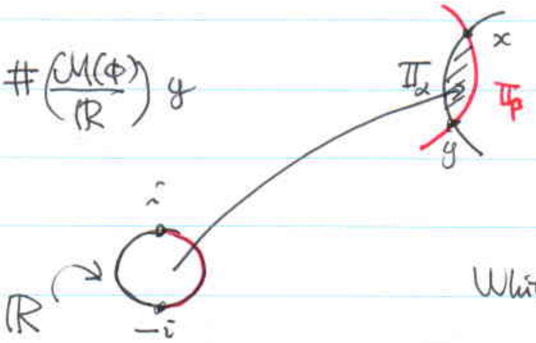
$$\widehat{CF}(Y) := \bigoplus_{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} (\mathbb{Z}/2\mathbb{Z})x$$

NB. $\underset{\text{alg. intersect. \#}}{\#\mathbb{T}_\alpha \cap \mathbb{T}_\beta} = \det(\#\alpha_i \cap \beta_j)_{i,j} = |H_1(Y;\mathbb{Z})|$

· differential

$$\partial x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ n_z(\phi)=0}} \#\left(\frac{\mathcal{M}(\phi)}{\mathbb{R}}\right) y$$

$\mathcal{M}(\phi)$ moduli space of holo. disks



Whitney disk

$\#(\phi \cap V_z) = n_z(\phi)$

$\left(V_z = z \times \mathrm{Sym}^{g-1}(\Sigma) \hookrightarrow \mathrm{Sym}^g \Sigma \right.$ cpx codim 1 subvar. $\left.\right)$

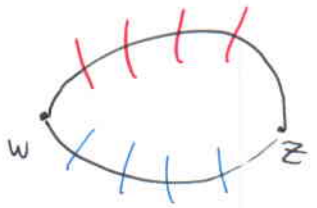
Fact $\partial^2 = 0$

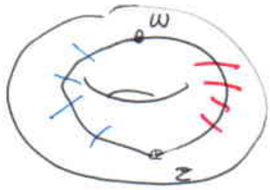
Thm (O-Szabó)

$H_*(\widehat{CF}(Y), \partial) = \widehat{HF}(Y)$ is a closed 3-mfd invariant.

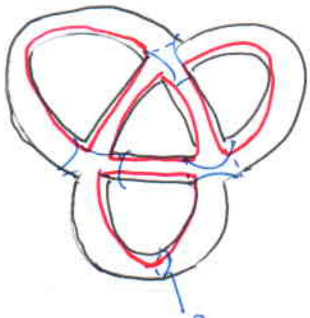
conjecturally $\cong$ Seiberg-Witten Floer homology

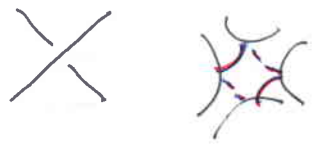
Choose w





$$\left(\Sigma, \begin{matrix}\alpha_1 \cdots \alpha_g \\ \beta_1 \cdots \beta_g\end{matrix}, w, z\right) \longrightarrow \text{knot } K \hookrightarrow \Sigma \ \begin{matrix}\alpha_1 \cdots \alpha_g \\ \beta_1 \cdots \beta_g\end{matrix}$$





$$\widehat{CFK}(Y) = \bigoplus_{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} (\mathbb{Z}/2\mathbb{Z})x$$

In the def. of ∂ we impose $n_w(\phi) = 0$

Thm (O-Szabó, Rassmusen)

$H_*(\widehat{CFK}(Y, K)) = \widehat{HFK}(K)$

is a knot inv.

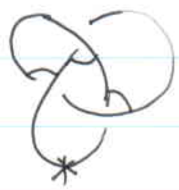
bigraded

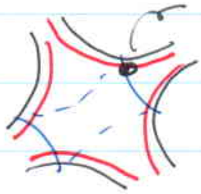
$A(x,y) := n_z(\phi) - n_w(\phi) \qquad \phi \in \pi_2(x,y)$

Observe $A(x,y)$ depends only x, y

$A(x,y) + A(y,w) = A(x,w)$

$\Rightarrow A(x,y) \overset{\exists}{=} A(x) - A(y)$





4 choices of $\cap$ points

$\rightsquigarrow$ Kauffman states for Alexander polynomial

combinatorial approaches

① Manolescu - O - Sarkar Summer 06

② Sarkar - Wang Summer 06

③ O - Szabó , O - Stipsicz - Szabó Summer 07